Application of Hodograph Method to Calculation of Time of Transfer in a Central Gravitational Field

M. N. Burdaev
Gagarin Cosmonaut Training Center, Zvyozdnny Gorodok, Moscow oblast, 141160 Russia
E-mail: bmn@starcity.ru
Received December 4, 2006

PACS: 91.10.Sp
DOI: 10.1134/S0010952509200099

In order to determine the time of transfer between two points of a central gravitational field, one can use several versions of known formulas derived on the basis of the Kepler equation [1, 2]. The initial equation for derivation of all these versions has the form:

\[ \Delta M = (E_N - E_M) - (e \sin E_N - e \sin E_M), \]  

(1)

where \( \Delta M \) is the difference of mean anomalies of the initial and final points of a transfer trajectory, \( E_M \) and \( E_N \) are eccentric anomalies of the initial and final points of the transfer trajectory, and \( e \) is eccentricity of the transfer trajectory.

In Eq. (1) and further on, orbit parameters are designated by subscripts \( M \) and \( N \) at the initial and final points of transfer, respectively.

Let us represent equation (1) in the form, in which the radii of boundary points of transfer \( r_M \) and \( r_N \), the angle between them \( \Delta \vartheta \), and angle \( \psi_M \) between the local vertical and orbital velocity vector at the initial point of transfer are independent variables (Fig. 1). For this purpose, we first investigate the main properties of transfer trajectories.

For the boundary points of transfer we have

\[ r_M = \frac{p}{1 + e \cos \vartheta_M}, \]  

(2)

\[ r_N = \frac{p}{1 + e \cos(\vartheta_M + \Delta \vartheta)}. \]  

(3)

Let us solve this system of equations for the eccentricity \( e \):

\[ e = \frac{1 - r_M}{r_N} \left( \frac{r_M}{r_N} - \cos \Delta \vartheta \right) \cos \vartheta_M + \sin \Delta \vartheta \sin \vartheta_M \]  

(4)

Then we substitute (4) into the known formula

\[ \cot \psi_M = \frac{e \sin \vartheta_M}{1 + e \cos \vartheta_M} = \frac{1 - \frac{r_M}{r_N}}{(1 - \cos \Delta \vartheta) \cot \vartheta_M + \sin \Delta \vartheta} \]

and determine from here \( \tan \vartheta_M \):

\[ \tan \vartheta_M = \frac{1 - \cos \Delta \vartheta}{(1 - \frac{r_M}{r_N}) \tan \psi_M - \sin \Delta \vartheta}. \]  

(5)

Having substituted quantities \( e \) from (3) and \( \tan \vartheta_M \) from (5) into Eq. (2), after some transformations we obtain:

\[ p = \frac{r_M}{(r_M/r_N) - \cos \Delta \vartheta \sin \Delta \vartheta} + \cot \psi_M \]  

(6)

Addressing to Fig. 1, it is possible to show that

\[ \frac{(r_M/r_N) - \cos \Delta \vartheta}{\sin \Delta \vartheta} = -\cot \psi_M, \]  

(7)

where \( \Delta \psi_M \) is the external angle of triangle \( OMN \) at vertex \( M \) determining the direction from initial point of transfer \( M \) to its final point \( N \).

Let us solve the known formula

\[ p = \frac{C^2}{\mu} = \frac{r^2 V^2 \sin^2 \psi}{\mu} \]  

185
From (8) we obtain the generalized equation for the hodograph of the initial velocity of transfer between two points of a central gravitational field:

$$\sqrt{\frac{r_M}{\mu}} V_M = \sqrt{\frac{\tan \frac{\Delta \theta}{2} (1 + \cot^2 \psi_M)}{\cot \psi_M - \cot \Delta \psi_M}}.$$  \hspace{1cm} (9)

In even more general form, this equation has the form:

$$\sqrt{\frac{r_M}{\mu}} \cot \frac{\Delta \theta}{2} V_M = \sqrt{\frac{\tan \psi_M + \cot \psi_M}{1 - \cot \Delta \psi M \tan \psi_M}}.$$  \hspace{1cm} (10)

There are only two variables, angles $\psi_M$ and $\Delta \psi_M$, in the right-hand side of this equation. Equation (10) for the first time has allowed one to show the general solution to the problem of calculating the initial velocity of transfer between two points of a central gravitational field in the two-dimensional form on a plane (Fig. 2) [3].

Let us reveal a connection between directions of the orbital velocity at initial ($M$) and final ($N$) points of the transfer trajectory. For this purpose, we equate each other the angular momentum values at these points.

$$r_M V_M \sin \psi_M = r_N V_N \sin \psi_N.$$  \hspace{1cm} (11)

From Eq. (9) it follows:

$$V_M \sin \psi_M = \sqrt{\frac{\mu}{r_M \cot \psi_M - \cot \Delta \psi_M}},$$  \hspace{1cm} (12)

$$V_N \sin \psi_N = \sqrt{\frac{\mu}{r_N \cot \psi_N - \cot \Delta \psi_N}},$$

where $\Delta \psi_N$ is the angle between a continuation of the line connecting points $M$ and $N$ and the direction from point $N$ to nadir. It is equal to the angle between the local vertical at point $N$ and the direction from it to point $M$ (Fig. 3). In accordance with (8), we have

$$\cot \Delta \psi_N = \frac{r_N - \cos \Delta \theta}{\sin \Delta \theta}.$$  \hspace{1cm} (13)

Having substituted (12) and (13) into (11) and having performed some transformations, we obtain

$$\cot \psi_N = -\frac{r_N}{r_M} \cot \psi_M + \left(\frac{r_N}{r_M} - 1\right) \cot \frac{\Delta \theta}{2}.$$  \hspace{1cm} (14)

Let us pass now to transformations of original equation (1).
We reduce the difference of eccentric anomalies of final $E_N$ and initial $E_M$ points of the transfer trajectory in the right-hand side of Eq. (1) to the following form:

$$E_N - E_M = 2 \arctan \left( \frac{\frac{1 - e}{1 + e} \tan \Delta \vartheta + \Delta \vartheta}{2} \right)$$

$$- \arctan \left( \frac{\frac{1 - e}{1 + e} \tan \Delta \vartheta}{2} \right)$$

$$= 2 \arctan \frac{\sqrt{1 - e^2}}{e \sin \vartheta_M \left( \tan \psi_M \cot \frac{\Delta \vartheta}{2} - 1 \right)}.$$  \hspace{1cm} (15)

It is known that

$$e = \sqrt{1 - \frac{r V^2}{\mu} \left( 2 - \frac{r V^2}{\mu} \right) \sin^2 \psi}.$$  

Hence

$$\sqrt{1 - e^2} = \sqrt{\frac{r V^2}{\mu} \left( 2 - \frac{r V^2}{\mu} \right)} \cot^2 \psi.$$  \hspace{1cm} (16)

Using (9), we rewrite (16) in the form:

$$\sqrt{1 - e^2} = \sqrt{\frac{\tan \frac{\Delta \vartheta}{2}}{\cot \psi_M - \cot \Delta \psi_M} \left[ 2 - \frac{\tan \frac{\Delta \vartheta}{2} (1 + \cot^2 \psi_M)}{\cot \psi_M - \cot \Delta \psi_M} \right]}.$$  \hspace{1cm} (17)

Having divided (17) by (18), we obtain

$$\sqrt{1 - e^2} = \frac{2 (1 - \cot \Delta \psi_M \tan \psi_M) \cot \frac{\Delta \vartheta}{2} \tan \frac{\Delta \vartheta}{2}}{e \sin \vartheta_M}$$

and substitute the obtained expression into (15):

$$E_N - E_M = 2 \arctan \left( \sqrt{\frac{2 (\cot \psi_M - \cot \Delta \psi_M) \cot \frac{\Delta \vartheta}{2} - \cot^2 \psi_M - 1}{\cot \frac{\Delta \vartheta}{2} - \cot \psi_M}} \right).$$  \hspace{1cm} (19)

Then we reduce the product $e \sin E$ to the form:

$$e \sin E = \frac{e \sqrt{1 - e^2} \sin \vartheta}{1 + e \cos \vartheta} = \sqrt{1 - e^2} \cot \psi.$$  

Using this expression, we can present the difference of products $e \sin E_N - e \sin E_M$ in the right-hand side of Eq. (1) as:

$$e \sin E_N - e \sin E_M = \sqrt{1 - e^2} (\cot \psi_N - \cot \psi_M).$$  \hspace{1cm} (20)

Equation (14) allows us to write:

$$\cot \psi_N - \cot \psi_M = \left( \frac{r_N}{r_M} - 1 \right) \cot \frac{\Delta \vartheta}{2} - \left( \frac{r_N}{r_M} + 1 \right) \cot \psi_M.$$

For the sake of convenience of further transformations, we reduce the last expression to the form:
\[
\cot \psi_N - \cot \psi_M = \left( \frac{r_N}{r_M} + 1 \right) \left( \cot \frac{\Delta \varnothing}{2} - \cot \psi_M \right) - 2 \cot \frac{\Delta \varnothing}{2}. \tag{21}
\]

From triangle \( OMN \) we determine (Fig. 1):
\[
\frac{r_N}{r_M} = \frac{1}{\cos \Delta \varnothing - \cot \Delta \psi_M \sin \Delta \varnothing}.
\]

With this substitution, relation (21) takes the form:
\[
\cot \psi_N - \cot \psi_M = \frac{\cot \frac{\Delta \varnothing}{2} - \cot \psi_M}{\cot \Delta \varnothing - \cot \Delta \psi_M} \left( \cot \frac{\Delta \varnothing}{2} - \cot \psi_M \right) - 2 \cot \frac{\Delta \varnothing}{2}. \tag{22}
\]

Substituting (17) and (22) into (20), we get after some simplification:
\[
\frac{e \sin E_N - e \sin E_M}{\sqrt{2 (\cot \psi_M - \cot \Delta \psi_M) \cot \frac{\Delta \varnothing}{2} - \cot^2 \psi_M - 1}}
\]
\[
\times \left[ \frac{\cot \frac{\Delta \varnothing}{2} - \cot \Delta \psi_M}{\cot \Delta \varnothing - \cot \Delta \psi_M} \left( 1 - \tan \frac{\Delta \varnothing}{2} \cot \psi_M \right) - 2 \right]. \tag{23}
\]

Having substituted (19) and (23) into (1), we obtain the equation for calculating the difference of mean anomalies of boundary points of transfer \( M \) and \( N \) as a function of three angles \( \Delta \varnothing, \Delta \psi_M \), and \( \psi_M \).

\[
\Delta M_{MN} = 2 \arctan \left[ \frac{\sqrt{2 (\cot \psi_M - \cot \Delta \psi_M) \cot \frac{\Delta \varnothing}{2} - \cot^2 \psi_M - 1}}{\cot \frac{\Delta \varnothing}{2} - \cot \psi_M} \right]
\]
\[
\times \left[ \frac{\cot \frac{\Delta \varnothing}{2} - \cot \Delta \psi_M}{\cot \Delta \varnothing - \cot \Delta \psi_M} \left( \tan \frac{\Delta \varnothing}{2} \cot \psi_M - 1 \right) + 2 \right]. \tag{24}
\]

Singular advantage of Eq. (24) consists in the following specific property: angles \( \Delta \varnothing \) and \( \Delta \psi_M \) uniquely determine a family of similar triangles formed by the boundary radii of transfer \( r_M \) and \( r_N \), and by angle between them \( \Delta \varnothing \), and at a constant angle of the initial velocity of transfer \( \psi_M \) they also determine a corresponding family of transfer orbits with an identical difference of mean anomalies of boundary points, \( \Delta M \).

In order to calculate, using this formula, the time of transfer between two points of central gravitational field we use the known relation
\[
\Delta M_{MN} = (t_N - t_M) \frac{\sqrt{\mu}}{a^2}. \tag{25}
\]

To conserve, when calculating, the unity and minimum quantity of variables, we present the semimajor axis of transfer trajectory \( a \) as a function of the same three angles and initial radius \( r_M \) of the transfer trajectory. For this purpose, we substitute into the known equation of the semimajor axis the quantity of generalized parameter of velocity from (9):
\[
a = \frac{r}{2 - \frac{r V^2}{\mu}} = \frac{r_M}{\tan \frac{\Delta \varnothing}{2} (1 + \cot^2 \psi_M) - \frac{2}{\cot \psi_M - \cot \Delta \psi_M}}. \tag{26}
\]

Having solved (25) for \( t_N - t_M \) and having substituted (26) into it, we obtain:
\[
t_N - t_M = \frac{1}{\sqrt{\mu}} \left[ \frac{r_M}{2 - \frac{r V^2}{\mu} \tan \frac{\Delta \varnothing}{2} (1 + \cot^2 \psi_M) - \frac{2}{\cot \psi_M - \cot \Delta \psi_M}} \right]^{3/2} \Delta M_{MN}. \tag{27}
\]

Equation (27) represents the dependence of the time of transfer along elliptical trajectories between two points of a central gravitational field on three angles, \( \Delta \varnothing, \Delta \psi_M \), and \( \psi_M \), determining a certain family of the orbits of transfer with an identical difference of mean anomalies \( \Delta M \) at boundary points, and on the radius of the initial point of transfer \( r_M \) playing the role of a time scale coefficient, when a particular trajectory of this family is selected.

At given boundary radii \( r_M \) and \( r_N \), angle between them \( \Delta \varnothing \), and time of transfer \( t_N - t_M \), determination of the orbit corresponding to them is carried out by the method of successive approximations with varying angle \( \psi_M \). After reaching a given accuracy of iterative calculations, the value of orbital velocity \( V_M \) is determined from Eq. (8), and further planar orbit elements...
are calculated by known methods using the values of $r_M$, $V_M$, and $\psi_M$.

To present day, the Lambert equation is used for iterative calculations in this algorithm. In the general case it has four variants of the solution; the selection of a variant for calculations depends on position of the transfer trajectory and its boundary radii relative to both orbit foci. It is noted in [4] that, when performing astronomical calculations, only two of these variants are used. The necessity to perform operations selecting the variant of the Lambert equation used for calculations complicates the algorithm and increases computer time, when orbits are determined with its application.

When determining orbits using Eq. (27), there is no need in these operations. This peculiarity of the new equation is a consequence of changed variable: the angle of velocity at the initial point of transfer is used in it instead of the semimajor axis of orbit commonly used in the Lambert equation. The numerical checking of Eq. (27) has shown that for it there is a single condition of verification and correction: if in the process of calculation the value of $E_N - E_M$ becomes less than zero, it is necessary to add $2\pi$ to the right-hand side of Eqs. (24) and (27).

For making iterative calculations according to formulas (24) or (27), one can use quantity $\psi_M$ or $\cot \psi_M$ as a variable. The use of quantity $\cot \psi_M$ is more preferable, because, in this case, in an iterative cycle, in contrast to the Lambert equation, it is not required to calculate direct trigonometric functions. This property of the new equation essentially reduces the duration of iterative cycle and the total duration of calculations.

The range of possible values of variables $\psi_M$ or $\cot \psi_M$ for elliptical orbits of transfers is determined using formula (10). The boundaries of this range for elliptical orbits correspond to parabolic orbits, for which the following relation takes place

$$\sqrt{\frac{r_M}{\mu}} V_M = \sqrt{2}.$$  

Having used this relation, from Eq. (10) we obtain

$$\sqrt{2 \cot \frac{\Delta \vartheta}{2}} = \frac{1 + \cot^2 \psi_{M,1,2}}{\cot \psi_{M,1,2} - \cot \Delta \psi_M},$$  

where $\psi_{M,1,2}$ are the boundaries of the range of angles $\psi_M$ for elliptical orbits of transfers.

Having solved Eq. (28) for $\cot \psi_{M,1,2}$, we find:

$$\cot \psi_{M,1,2} = \cot \frac{\Delta \vartheta}{2} \pm \sqrt{\left(\cot \frac{\Delta \vartheta}{2} - 2 \cot \Delta \psi_M\right) \cot \frac{\Delta \vartheta}{2} - 1}.$$  

Upon substitution of quantity $\cot \Delta \psi_M$ from (28) into this formula, we obtain

$$\cot \psi_{M,1,2} = \cot \frac{\Delta \vartheta}{2} \pm \frac{r_M}{r_N} \left(1 + \cot^2 \Delta \vartheta\right).$$  

As an example, Fig. 4 shows the range of angles of velocity $\psi_M$ at the initial point of transfers along elliptical orbits for the relation of boundary radii of transfers $r_M/r_N = 0.5$ calculated using Eq. (29).

The part of the range of values $\cot \psi_M$, corresponding to these angles $\psi_M$ for values $\Delta \vartheta = 50^\circ - 310^\circ$ is shown in Fig. 5.

To select the initial value $\cot \psi_M$ or $\psi_M$ in iterative calculations we determine parameters of a transfer.
mode with minimum initial velocity. Having taken the first derivative of parameters of the initial velocity of transfer from Eqs. (8), (9), or (10), we find that angle \( \psi_{\text{avmin}} \) between the vector of minimum initial velocity of transfer and the local vertical at point \( M \) is equal to a half of angle \( \Delta \psi_{\text{av}} \):

\[
\psi_{\text{avmin}} = \Delta \psi_{\text{av}} / 2.
\]

For angle \( \psi_{\text{avmin}} \) using Eq. (27), we calculate the time of transfer \( \Delta t_{MN} \) between boundary points \( M \) and \( N \) corresponding to the minimum initial velocity of transfer. Comparing this time with specified in initial conditions time of transfer \( \Delta t_{1-2} \), we determine to what side along \( \psi_{M} \) one should move in the iterative process. At \( \Delta t_{MN} > \Delta t_{1-2} \), the step in \( \psi_{M} \) should be chosen so that the value of \( \psi_{M} \) would increase. At \( \Delta t_{MN} < \Delta t_{1-2} \), the step in \( \psi_{M} \) should be chosen having in mind decreasing value of \( \psi_{M} \).

The method presented in this paper allows one to derive similar equations for the time of transfer between two points of a central gravitational field for parabolic and hyperbolic trajectories as well.

**REFERENCES**